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ON THE GROWTH OF ENTIRE FUNCTIONS

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ABSTRACT

Suppose that $\alpha > 1$, $0 < R < \infty$ and that f is analytic in $|z| \le \alpha R$ with $|f(0)| \ge 1$. It is shown that for a constant d_α depending only on α,

$$\log M(R,f) \le d_\alpha T(R,f)^{1/2} T(\alpha R,f)^{1/2}.$$

Therefore if f is entire of order $\lambda < \infty, \log M(r,f)/T(r,f)$ has order at most $\lambda/2$. These results are shown by example to be quite precise.

1. Introduction

Let $f(z)$ be meromorphic in the complex plane. We will use freely the standard notation of Nevanlinna theory, including

$$T(r,f),\ m(r,f),\ N(r,f),\ \log M(r,f), \ldots .$$

In addition, we define $m_p^+(r,f)$, $1 < p < \infty$, by

$$m_p^+(r,f) = \left[\frac{1}{2\pi} \int_0^{2\pi} (\log^+ |f(re^{i\theta})|)^p d\theta \right]^{1/p} .$$

It has long been of interest to compare the sizes of $T(r,f)$ and $\log M(r,f)$ for entire functions. In 1932 R.E.A.C. Paley conjectured that an entire function $f(z)$ of order λ satisfies

$$\varliminf_{r\to\infty} \frac{\log M(r,f)}{T(r,f)} \le \begin{cases} \frac{\pi\lambda}{\sin \pi\lambda}, & \lambda \le \frac{1}{2}, \\ \pi\lambda, & \lambda > \frac{1}{2}. \end{cases} \tag{1.1}$$

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This conjecture was proved by Valiron [10] and Wahlund [11] for $\lambda < \frac{1}{2}$ in 1935. The first complete proof was given by Govorov [3] in 1969. Petrenko [7] has established that the inequality (1.1) remains valid if the order λ is replaced by the lower order μ and $f(z)$ is assumed to be meromorphic.

The situation is quite different for entire functions of infinite order. In fact, for such functions

$$T(r,f) = o(\log M(r,f)), \quad r \to \infty,$$

is possible. An upper bound for $\log M(r,f)$ in terms of $T(r,f)$ was obtained by Shimizu [9] in 1929 when he showed that for each number $k > 1$ and each entire function $f(z)$

$$\varliminf_{r\to\infty} \frac{\log M(r,f)}{T(r,f)[\log T(r,f)]^k} = 0.$$

Related results can be found in papers by C. T. Chuang [2], I.I. Marchenko and A. I. Shcherba [6], W. Bergweiler [1], and A. I. Shcherba [8].

We note that all the upper bounds mentioned above have exceptional sets. Less attention seems to have been given to obtaining upper bounds for $\log M(r,f)$ in terms of $T(r,f)$ having no exceptional set. From the results of W. K. Hayman in [5] it follows that for any increasing function $g(t)$ there exists an entire function $f(z)$ such that

$$\varlimsup_{r\to\infty} \frac{\log M(r,f)}{g(T(r,f))} = \infty.$$

This means that we cannot estimate $\log M(r,f)$ from above in terms of $T(r,f)$ for all $r \in (0,\infty)$ for the class of all entire functions. For entire functions of finite order, A. I. Shcherba [8] obtained in 1985 that for any number $k > 0$,

$$\lim_{r\to\infty} \frac{\log M(r,f)}{\exp\{kT(r,f)\}} = 0.$$

In this paper, we show that for an entire function $f(z)$ of order $\lambda < \infty$ and $r \geq r(\varepsilon)$,

$$\log M(r,f) \leq r^{\frac{\lambda}{2}+\varepsilon} T(r,f), \quad \varepsilon > 0. \tag{1.2}$$

Inequality (1.2) follows immediately from Theorem 2.2, which asserts that

$$\log M(r,f) \leq d_\alpha T(r,f)^{\frac{1}{2}} T(\alpha r,f)^{\frac{1}{2}}, \tag{1.3}$$

for all entire f with $|f(0)| \geq 1$ and all $r > 0$, where $\alpha > 1$ and d_α is a constant depending only on α. The inequality (1.3) improves the well-known inequality

$$\log M(r,f) \leq \frac{R+r}{R-r} T(R,f), \quad 0 < r < R, \tag{1.4}$$

which follows from the Poisson–Jensen formula. All the facts we mentioned above have analogues for $m_p^+(r,f)$.

In section 3 we construct examples, using a technique which was introduced by W. K. Hayman [5] in his study of the comparative sizes of $T(r,f)$ and $T(r,f')$ for meromophic functions and also used by A. I. Shcherba [8], to show all the results of section 2 are best possible.

2. Main Results

In this section we obtain upper bounds for $\log M(r,f)$ in terms of the Nevanlinna characteristic for entire functions with no exceptional set of r. We first need the following estimate for $\partial \log |f(re^{i\theta})|/\partial r$.

LEMMA 2.1: *Let $1 < \beta < e$ and let $0 < R < \infty$. Suppose that $f(z)$ is analytic in $|z| \leq \beta^2 R$ with $|f(0)| \geq 1$, that $f(z)$ has no zeros in $|z| < R$, and that $0 \leq \theta < 2\pi$. Then for $R/2 \leq r < R$, we have*

$$\frac{\partial \log |f(re^{i\theta})|}{\partial r} \leq c_\beta \frac{T(\beta^2 R, f)}{R},$$

where

$$c_\beta = \frac{6\beta}{(\beta-1)^2}.$$

Proof: Let a_n be the zeros of $f(z)$ and let $w = re^{i\theta}, R/2 \leq r < R$, and $0 \leq \theta < 2\pi$. Without loss of generality we may assume $|a_n| \neq \beta R$ for all n. The differentiated Poisson–Jensen formula [4, p. 22] gives

$$\begin{aligned}\frac{wf'(w)}{f(w)} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(\beta Re^{i\varphi})| \frac{2w\beta Re^{i\varphi}}{(\beta Re^{i\varphi} - w)^2} d\varphi \\ + \sum_{|a_n|<\beta R} \left(\frac{w}{w-a_n} + \frac{\bar{a_n} w}{(\beta R)^2 - \bar{a_n} w}\right).\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{Re}\left[\frac{wf'(w)}{f(w)}\right] \leq & \frac{2r\beta R}{(\beta R - r)^2}\left(\frac{1}{2\pi}\int_0^{2\pi} |\log|f(\beta Re^{i\varphi})||d\varphi\right) \\ & + \sum_{|a_n|<\beta R}\left[\mathrm{Re}\left(\frac{w}{w-a_n}\right) + \frac{|a_n| r}{(\beta R)^2 - |a_n| r}\right].\end{aligned} \tag{2.1}$$

Since $\xi = z/(z-a)$ maps the circle $|z| = |a|$ to the line $\mathrm{Re}(\xi) = \frac{1}{2}$, we have for $|a| > r = |w|$ that

$$\mathrm{Re}\left[\frac{w}{w-a}\right] < \frac{1}{2}.$$

By assumption, $f(z)$ has no zeros in $|z| < R$. Therefore $|a_n| \geq R > r$ for all $n = 1, 2, \ldots$, and we have

$$\mathrm{Re}\left[\frac{w}{w-a_n}\right] < \frac{1}{2}. \tag{2.2}$$

It is trivial to show for $|a_n| < \beta R$ that

$$\frac{|a_n| r}{(\beta R)^2 - |a_n| r} \leq \frac{r}{\beta R - r}. \tag{2.3}$$

By assumption, we have $\log|f(0)| \geq 0$. Therefore Jensen's formula gives

$$T(\beta R, 1/f) = T(\beta R, f) - \log|f(0)| \leq T(\beta R, f).$$

Hence we obtain that

$$\frac{1}{2\pi}\int_0^{2\pi} |\log|f(\beta R e^{i\varphi})||d\varphi \leq 2T(\beta R, f). \tag{2.4}$$

Observing that

$$r\frac{\partial \log|f(re^{i\theta})|}{\partial r} = \mathrm{Re}\left[\frac{re^{i\theta} f'(re^{i\theta})}{f(re^{i\theta})}\right],$$

we deduce from (2.1), (2.2), (2.3), and (2.4) that

$$\frac{\partial \log|f(re^{i\theta})|}{\partial r} \leq \frac{4\beta R}{(\beta R - r)^2} T(\beta R, f) + \left(\frac{1}{2r} + \frac{1}{\beta R - r}\right) n(\beta R, 1/f).$$

Since $\log\beta \geq \frac{\beta - 1}{2}$ for $1 < \beta < e$, we have

$$n(\beta R, 1/f) \leq \frac{N(\beta^2 R, 1/f)}{\log\beta} \leq \frac{2T(\beta^2 R, f)}{\beta - 1}.$$

Hence we finally obtain that

$$\begin{aligned}\frac{\partial \log|f(re^{i\theta})|}{\partial r} &\leq \frac{4\beta}{(\beta-1)^2}\left(\frac{T(\beta R, f)}{R}\right) + \left(\frac{1}{R} + \frac{1}{(\beta-1)R}\right)\frac{2T(\beta^2 R, f)}{\beta - 1} \\ &\leq \left(\frac{6\beta}{(\beta-1)^2}\right)\frac{T(\beta^2 R, f)}{R}.\end{aligned}$$

This proves the lemma.

We now use Lemma 2.1 to obtain our main results.

THEOREM 2.2: *Suppose that $1 < \alpha < e^2$ and $0 < R < \infty$. Let $f(z)$ be analytic in $|z| \le \alpha R$ with $|f(0)| \ge 1$, and let $1 < p < \infty$. Then we have*

$$\log M(R,f) \le d_\alpha T(R,f)^{\frac{1}{2}} T(\alpha R,f)^{\frac{1}{2}} \tag{2.5}$$

and

$$m_p^+(R,f) \le d_\alpha^{\frac{p-1}{p}} T(R,f)^{\frac{p+1}{2p}} T(\alpha R,f)^{\frac{p-1}{2p}} \tag{2.6}$$

where

$$d_\alpha = \frac{4\sqrt{3}\alpha^{\frac{1}{2}}(\alpha^{\frac{1}{2}}+1)}{\alpha-1}.$$

We show that the results of Theorem 2.2 are quite precise in Theorem 3.2 (a) and (b).

Proof:
(i) Suppose first that $f(z)$ has no zeros in $|z| < R$. We choose θ_0 in $[0, 2\pi)$ so that

$$\log M(R,f) = \log|f(Re^{i\theta_0})| > 0.$$

For $\frac{R}{2} \le R_1 < R$, we use Lemma 2.1 (with the choice $\beta^2 = \alpha$) and (1.4) to obtain

$$\begin{aligned} \log M(R,f) &= \log|f(R_1 e^{i\theta_0})| + \int_{R_1}^{R} \frac{\partial \log|f(re^{i\theta_0})|}{\partial r} dr \\ &\le \log M(R_1,f) + \int_{R_1}^{R} c_\beta \frac{T(\beta^2 R,f)}{R} dr \\ &\le \frac{2R}{R-R_1} T(R,f) + (R-R_1) c_\beta \frac{T(\beta^2 R,f)}{R}. \end{aligned} \tag{2.7}$$

Now choose R_1 such that

$$R - R_1 = \frac{\log M(R,f)}{c_\beta T(\beta^2 R,f)} \left(\frac{R}{2}\right). \tag{2.8}$$

We claim that $R_1 \ge \frac{R}{2}$. In fact, by (1.4)

$$\frac{\log M(R,f)}{c_\beta T(\beta^2 R,f)} \le \frac{(\beta-1)^2(\beta^2+1)}{6\beta(\beta^2-1)} = \frac{(\beta-1)(\beta^2+1)}{6\beta(\beta+1)} < \frac{e^2+1}{12} < 1$$

since $1 < \beta < e$. This establishes our claim. Hence it follows from (2.7) and (2.8) that

$$\log M(R,f) \le 4c_\beta \frac{T(R,f)T(\beta^2 R,f)}{\log M(R,f)} + \frac{1}{2}\log M(R,f).$$

Therefore we conclude

$$\log M(R,f) \le (8c_\beta)^{\frac{1}{2}} T(R,f)^{\frac{1}{2}} T(\beta^2 R,f)^{\frac{1}{2}}.$$

Since $\alpha = \beta^2$, we get

$$\log M(R,f) \le d_\alpha T(R,f)^{\frac{1}{2}} T(\alpha R,f)^{\frac{1}{2}},$$

where

$$d_\alpha = \frac{4\sqrt{3}\alpha^{\frac{1}{2}}(\alpha^{\frac{1}{2}}+1)}{\alpha - 1}.$$

(ii) In the general case, if a_n are the zeros of $f(z)$, set

$$g(z) = f(z) \prod_{|a_n|<R} \frac{R^2 - \bar{a_n} z}{R(z-a_n)}.$$

Then $g(z)$ has no zeros in $|z| < R$, and $|g(0)| \ge 1$. We may apply the reasoning of (i) to the function $g(z)$ to obtain

$$\log M(R,g) \le d_\alpha T(R,g)^{\frac{1}{2}} T(\alpha R,g)^{\frac{1}{2}}. \tag{2.9}$$

Since

$$|g(z)| = |f(z)| \quad \text{on } |z| = R$$

and

$$|g(z)| \le |f(z)| \quad \text{on } |z| = \alpha R,$$

we have

$$\log M(R,g) = \log M(R,f), \tag{2.10}$$

$$T(R,g) = T(R,f), \tag{2.11}$$

and

$$T(\alpha R,g) \le T(\alpha R,f). \tag{2.12}$$

Hence it follows immediately from (2.9), (2.10), (2.11), and (2.12) that

$$\log M(R,f) \le d_\alpha T(R,f)^{\frac{1}{2}} T(\alpha R,f)^{\frac{1}{2}}.$$

This proves (2.5).

(iii) Finally, a simple observation gives (2.6). In fact,

$$\begin{aligned}[m_p^+(R,f)]^p &= \frac{1}{2\pi}\int_0^{2\pi}(\log^+|f(Re^{i\theta})|)^p d\theta \\ &\le \frac{1}{2\pi}\int_0^{2\pi}(\log M(R,f))^{p-1}\log^+|f(Re^{i\theta})|d\theta \\ &= \log M(R,f)^{p-1}T(R,f) \le d_\alpha^{p-1}T(R,f)^{\frac{p+1}{2}}T(\alpha R,f)^{\frac{p-1}{2}}.\end{aligned}$$

This establishes (2.6) and the proof of Theorem 2.2 is complete.

For entire functions of finite order, we now obtain upper bounds for $\log M(r,f)$ and $m_p^+(r,f)$ in terms of $T(r,f)$.

COROLLARY 2.3: *Let $f(z)$ be entire of order λ, $0<\lambda<\infty$. Then for any $\varepsilon>0$ there exists a number $R(\varepsilon)$ such that*

$$\log M(R,f) \le R^{\frac{\lambda}{2}+\varepsilon}T(R,f), \quad R>R(\varepsilon), \tag{2.13}$$

and

$$m_p^+(R,f) \le R^{\frac{(p-1)}{2p}\lambda+\varepsilon}T(R,f), \quad R>R(\varepsilon). \tag{2.14}$$

We show that the results of Corollary 2.3 are best possible in Theorem 3.2 (c).

Proof: Without loss of generality we may assume that $|f(0)|\ge 1$. If not, consider the function $f(z)+2$. By setting $\alpha=2$ in (2.5), we get

$$\log M(R,f) \le 4\sqrt{6}(\sqrt{2}+1)T(R,f)^{\frac12}T(2R,f)^{\frac12} \le 40T(2R,f)^{\frac12}T(R,f). \tag{2.15}$$

By the definition of order, we have

$$T(2R,F) = 0((2R)^{\lambda+\varepsilon}) = 0(R^{\lambda+\varepsilon}), \quad R\to\infty. \tag{2.16}$$

Hence (2.13) follows from (2.15) and (2.16).

Now, by setting $\alpha=2$ in (2.6), we get

$$\begin{aligned}m_p^+(R,f) &\le [4\sqrt{6}(\sqrt{2}+1)]^{\frac{p-1}{p}}T(R,f)^{\frac{p+1}{2p}}T(2R,f)^{\frac{p-1}{2p}} \\ &\le 40T(2R,f)^{\frac{p-1}{2p}}T(R,f).\end{aligned} \tag{2.17}$$

Hence (2.14) also follows from (2.16) and (2.17).

3. Examples

We now proceed to construct examples of entire functions which show that Theorem 2.2 and Corollary 2.3 are quite precise. Our examples are based on techniques introduced by Hayman [5]. We let

$$f(z) = f(z,c) = \exp\left(\frac{c}{1-z}\right), \quad c \geq 1.$$

Since $\mathrm{Re}[\frac{c}{1-z}]$ is positive and harmonic in $|z| < 1$, we have, for $0 < k < 1$,

$$T(k,f) = \log f(0) = c$$

and

$$\log M(k,f) = \frac{c}{1-k}.$$

We set

$$f(z,c) = \sum_{n=0}^{\infty} a_n z^n,$$

$$P_N(z,c) = \sum_{n=0}^{N} a_n z^n,$$

and

$$K_N = 1 - 8\left(\frac{2c}{N}\right)^{\frac{1}{2}}.$$

We need the following two lemmas of Hayman.

LEMMA A [5]: *If $K_N > \frac{1}{2}$ and $|z| = r$, then we have*

(a) $|P_N(z,c) - f(z,c)| < 1, \ r \leq K_N,$

(b) $|T(r, P_N(z,c)) - c| \leq \log 2, \ r \leq K_N,$

(c) $T(r, P_N(z,c)) \leq 17(cN)^{\frac{1}{2}} + N \log^+ r, \ r > K_N.$

We suppose that r_n, c_n, and N_n are increasing sequences satisfying the following conditions for all positive integers n:

(3.1) (i) $$r_1 = 1, \quad r_{n+1} \geq 2r_n,$$

(ii) $$c_n \geq n,$$

and

(iii) $$N_n \geq 512 c_n,$$

where the N_n are positive integers. We set

$$F(z)=\sum_{n=1}^{\infty} e^{-4c_n} P_{N_n}\left(\frac{z}{r_n}, c_n\right) \tag{3.2}$$

and have

LEMMA B [5]: *With the above notation $F(z)$ is an entire function. Further, if $r_\nu/2 \le r < r_{\nu+1}/2$, we have, for $|z| = r$ and $\nu \ge 2$,*

$$\text{(a)} \qquad T(r,F) \le T\left(\frac{r}{r_\nu}, e^{-4c_\nu} P_{N_\nu}(z, c_\nu)\right) + \sum_{n=1}^{\nu-1} N_n \left(\log\frac{r}{r_n} + 2\right),$$

and

$$\text{(b)} \qquad \Big| \sum_{n=\nu+1}^{\infty} e^{-4c_n} P_{N_n}\left(\frac{z}{r_n}, c_n\right) \Big| < 1.$$

In addition we need the following elementary lemma.

LEMMA 3.1: *If $1 < p < \infty$ and $0 < k < 1$, then*

$$m_p^+(k, f(z,c)) \ge \frac{c}{2\pi(1-k)^{\frac{p-1}{p}}}.$$

Proof: If $|\theta| \le 1-k$, then we have

$$\begin{aligned}\operatorname{Re}\left(\frac{1}{1-ke^{i\theta}}\right) &= \frac{1-k\cos\theta}{1+k^2-2k\cos\theta} \ge \frac{1-k}{(1-k)^2+2k(1-\cos\theta)} \\ &\ge \frac{1-k}{(1-k)^2+2k(\theta^2/2)} \ge \frac{1-k}{(1-k)^2+k(1-k)^2} \ge \frac{1}{2(1-k)}.\end{aligned}$$

Hence

$$\begin{aligned}[m_p^+(k, f(z,c))]^p &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\log^+\left|\exp\left(\frac{c}{1-ke^{i\theta}}\right)\right|\right]^p d\theta \\ &= \frac{c^p}{2\pi}\int_{-\pi}^{\pi}\left[\operatorname{Re}\left(\frac{1}{1-ke^{i\theta}}\right)\right]^p d\theta \ge \frac{c^p}{2\pi}2(1-k)\left[\frac{1}{2(1-k)}\right]^p \\ &= \frac{1}{\pi 2^p}c^p(1-k)^{1-p}.\end{aligned}$$

This proves the lemma.

Now we are prepared to prove

THEOREM 3.2: *Let $\varphi(r)$ be any positive increasing function such that*

$$\varphi(r) \to \infty, \quad r \to \infty,$$

and let $0 < \lambda < \infty$. Then there exists an entire function $F_1(z)$ such that

$$T(r, F_1) = 0(\varphi(r)(\log r)^2), \quad r \to \infty,$$

an entire function $F_2(z)$ of order λ and an entire function $F_3(z)$ of infinite order for which if $\varepsilon > 0$, $1 < \alpha < 2$, and $1 < p < \infty$, we have

(a) $$\overline{\lim_{r\to\infty}} \frac{\log M(r,F_i)}{T(r,F_i)^{\frac{1}{2}+\varepsilon}T(\alpha r,F_i)^{\frac{1}{2}-\varepsilon}} = \infty$$

and

$$\overline{\lim_{r\to\infty}} \frac{m_p^+(r,F_i)}{T(r,F_i)^{\frac{p+1}{2p}+\varepsilon}T(\alpha r,F_i)^{\frac{p-1}{2p}-\varepsilon}} = \infty, \quad i = 1,2,3;$$

(b) $$\overline{\lim_{r\to\infty}} \frac{\log M(r,F_i)}{T(\alpha r,F_i)} \geq \frac{1}{\alpha-1}(4-3\alpha)$$

and

$$\overline{\lim_{r\to\infty}} \frac{m_p^+(r,F_i)}{T(\alpha r,F_i)} \geq \left(\frac{1}{\alpha-1}\right)^{\frac{p-1}{p}} \left\{\frac{1}{2\pi} - 4(\alpha-1)^{\frac{p-1}{p}}\right\}, \quad i = 1,2,3;$$

(c) $$\overline{\lim_{r\to\infty}} \frac{\log M(r,F_2)}{T(r,F_2)r^{\frac{\lambda}{2}}} = \infty$$

and

$$\overline{\lim_{r\to\infty}} \frac{m_p^+(r,F_2)}{T(r,F_2)r^{\frac{p-1}{2p}\lambda}} = \infty.$$

Remarks:

(i) We observe that the conclusions in (b) are of interest only for values of α slightly greater than 1.

(ii) For entire functions $f(z)$ with

$$T(r,f) = 0(\log r)^2 \quad \text{as } r \to \infty, \tag{3.3}$$

W. K. Hayman has shown in [5] that

$$\lim_{r\to\infty} \frac{\log M(r,f)}{T(r,f)} = 1. \tag{3.4}$$

Parts (a) and (b) of Theorem 3.2, applied to F_1, show that in general (3.4) fails for entire functions not satisfying (3.3).

(iii) By considering F_1, F_2, and F_3, we see that inequalities (a) and (b) can occur for entire functions of any growth rate exceeding (3.3).

(iv) Part (a) shows that if in Theorem 2.2 we consider possible pairs of exponents for $T(R,f)$ and $T(\alpha R,f)$ with sum 1, then the choice of exponent $\frac{1}{2}$ on $T(\alpha R,f)$ in (2.5) and $\frac{p-1}{2p}$ on $T(\alpha R,f)$ in (2.6) cannot be improved.

(v) From part (a) we deduce that

$$\overline{\lim_{r\to\infty}} \frac{m_p^+(r,F_1)}{T(r,F_1)} = \infty,$$

which shows that even a weakened form of (3.4) obtained by replacing the numerator by $m_p^+(r,f)$ does not hold without the restriction (3.3).

(vi) In part (b), we see that the constant d_α of Theorem 2.2 is quite precise in the sense that (2.5) and (2.6) do not hold for any choice of d_α satisfying

$$d_\alpha = o\left(\frac{1}{\alpha-1}\right), \quad \alpha \to 1^+.$$

(vii) Part (c) shows that in Corollary 2.3 the exponents on R on the right side of (2.13) and (2.14) cannot be decreased.

Proof: Suppose that the sequences r_n, c_n and N_n satisfy (3.1) as well as

(3.5)(i) $$r_1 = 1, \quad r_{n+1} \geq 4r_n,$$

(ii) $$nN_{n-1}\log r_n = o(c_n), \quad n\to\infty,$$

and

(iii) $$\frac{N_n}{c_n} \to \infty, \quad \text{as } n\to\infty.$$

We also assume that $F(z)$ is constructed as in (3.2) with these new sequences. Later we will make specific choices of c_n, r_n, and N_n for each F_i, $1 \le i \le 3$, satisfying (3.5) and of course (3.1). Next we set, for all positive integers n,

$$R_n = r_n K_{N_n} \quad \text{and} \quad S_n = \frac{R_n}{\alpha},$$

where

$$K_{N_n} = 1 - 8\left(\frac{2c_n}{N_n}\right)^{\frac{1}{2}} > \frac{1}{2}.$$

Then we have

LEMMA 3.3: *With the above notation, for $r_\nu/2 \le r \le R_\nu$, $\nu \ge 2$, we have*

(a) $$\log M(r,F) \ge \log M\left(\frac{r}{r_\nu}, f(z,c_\nu)\right) - \{4+o(1)\}c_\nu, \quad \nu \to \infty,$$

and

(b) $$m_p^+(r,F) \ge m_p^+\left(\frac{r}{r_\nu}, f(z,c_\nu)\right) - \{4+o(1)\}c_\nu, \quad \nu \to \infty.$$

Proof: To prove this lemma, we suppose that $|z| \le R_\nu$ and $n \le \nu - 1$. Then by the inequality (1.4) and Lemma A (c) we obtain

$$\begin{aligned}\log\left|P_{N_n}\left(\frac{z}{r_n}, c_n\right)\right| &\le \log M\left(\frac{R_\nu}{r_n}, P_{N_n}(z,c_n)\right) \le 3T\left(2\frac{R_\nu}{r_n}, P_{N_n}(z,c_n)\right) \\ &\le 3\left\{17(c_nN_n)^{\frac{1}{2}} + N_n\log^+\left(2\frac{R_\nu}{r_n}\right)\right\} \le 3\{17N_{\nu-1} + N_{\nu-1}\log(2R_\nu)\} \\ &\le 60N_{\nu-1}\log R_\nu.\end{aligned}$$

Therefore

(3.6) $$\left|\sum_{n=1}^{\nu-1} e^{-4c_n} P_{N_n}\left(\frac{z}{r_n}, c_n\right)\right| \le (\nu-1)e^{60N_{\nu-1}\log R_\nu}.$$

Hence we deduce from Lemma A (a), Lemma B (b) and (3.6) that for $r_\nu/2 \le |z| \le R_\nu$,

$$|F(z)| \ge \left|e^{-4c_\nu} f\left(\frac{z}{r_\nu}, c_\nu\right)\right| - e^{-4c_\nu}\left|P_{N_\nu}\left(\frac{z}{r_\nu}, c_\nu\right) - f\left(\frac{z}{r_\nu}, c_\nu\right)\right|$$

(3.7)

$$- \left|\sum_{n\ne\nu} e^{-4c_n} P_{N_n}\left(\frac{z}{r_n}, c_n\right)\right| \ge \left|e^{-4c_\nu} f\left(\frac{z}{r_\nu}, c_\nu\right)\right| - \nu e^{60N_{\nu-1}\log R_\nu}.$$

From familiar properties of $\log^+$, we conclude from (3.5) (ii) and (3.7) that

$$\log^+|F(z)| \geq \left\{\log^+\left|f\left(\frac{z}{r_\nu}, c_\nu\right)\right| - 4c_\nu\right\} - \log^+\nu - 60N_{\nu-1}\log R_\nu - \log 2$$

(3.8)

$$= \log^+\left|f\left(\frac{z}{r_\nu}, c_\nu\right)\right| - (4+o(1))c_\nu, \quad \nu \to \infty.$$

The conclusion of Lemma 3.3 is immediate from (3.8).

We now estimate $T(R_\nu, F)$ and $T(\alpha R_\nu, F)$ from above and $\log M(R_\nu, F)$ and $m_p^+(R_\nu, F)$ from below. Recall that $R_\nu = K_{N_\nu} r_\nu > r_\nu/2$ and

$$K_{N_\nu} = 1 - 8\left(\frac{2c_\nu}{N_\nu}\right)^{\frac{1}{2}}. \tag{3.9}$$

Since by Lemma A (b)

$$T\left(\frac{R_\nu}{r_\nu}, e^{-4c_\nu}P_{N_\nu}(z, c_\nu)\right) \leq T(K_{N_\nu}, P_{N_\nu}(z, c_\nu)) \leq c_\nu + \log 2,$$

we deduce from Lemma B (a) and (3.5) (ii) that

$$T(R_\nu, F) \leq T\left(\frac{R_\nu}{r_\nu}, e^{-4c_\nu}P_{N_\nu}(z, c_\nu)\right) + \sum_{n=1}^{\nu-1} N_n\left(\log\frac{R_\nu}{r_n} + 2\right)$$

(3.10)

$$\leq (c_\nu + \log 2) + (\nu-1)N_{\nu-1}(\log R_\nu + 2) \leq (1+o(1))c_\nu, \quad \nu \to \infty.$$

Since by (3.5) (i)

$$\frac{r_\nu}{2} < R_\nu < \alpha R_\nu < 2r_\nu \leq \frac{r_{\nu+1}}{2},$$

we may choose $r = \alpha R_\nu$ in Lemma B (a) and then obtain from Lemma A (c) and (3.5) that

$$\begin{aligned} T(\alpha R_\nu, F) &\leq T(\alpha K_{N_\nu}, e^{-4c_\nu}P_{N_\nu}(z, c_\nu)) + \sum_{n=1}^{\nu-1} N_n\left(\log\frac{\alpha R_\nu}{r_n} + 2\right) \\ &\leq T(\alpha K_{N_\nu}, P_{N_\nu}(z, c_\nu)) + (\nu-1)N_{\nu-1}(\log r_\nu + \log\alpha + 2) \\ &\leq 17(c_\nu N_\nu)^{\frac{1}{2}} + N_\nu\log^+(\alpha K_{N_\nu}) + o(c_\nu) \\ &\leq (1+o(1))N_\nu\log\alpha, \quad \nu \to \infty. \end{aligned} \tag{3.11}$$

Recall that

$$\log M(K_{N_\nu}, f(z, c_\nu)) = \frac{c_\nu}{1 - K_{N_\nu}}. \tag{3.12}$$

Thus we deduce from Lemma 3.1, Lemma 3.3, (3.5) (iii), and (3.9) that, for $\nu \geq 2$,

$$\begin{aligned} \log M(R_\nu, F) &\geq \log M(K_{N_\nu}, f(z, c_\nu)) - \{4 + o(1)\}c_\nu \\ &= \frac{c_\nu}{1 - K_{N_\nu}} - \{4 + o(1)\}c_\nu = \left\{\frac{1}{8\sqrt{2}} - o(1)\right\} c_\nu^{\frac{1}{2}} N_\nu^{\frac{1}{2}}, \quad \nu \to \infty, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} m_p^+(R_\nu, F) &\geq m_p^+(K_{N_\nu}, f(z, c_\nu)) - \{4 + o(1)\}c_\nu \\ &\geq \frac{1}{2\pi}\left\{\frac{c_\nu}{(1 - K_{N_\nu})^{\frac{p-1}{p}}}\right\} - \{4 + o(1)\}c_\nu \\ &= \frac{1}{2\pi}\left(\frac{1}{8\sqrt{2}}\right)^{\frac{p-1}{p}} c_\nu^{\frac{p+1}{2p}} N_\nu^{\frac{p-1}{2p}} - \{4 + o(1)\}c_\nu \\ &\geq \left(\frac{1}{16\sqrt{2}\pi} - o(1)\right) c_\nu^{\frac{p+1}{2p}} N_\nu^{\frac{p-1}{2p}}, \quad \nu \to \infty. \end{aligned} \tag{3.14}$$

Now recall that $1 < \alpha < 2$ and $S_\nu = R_\nu/\alpha = r_\nu K_{N_\nu}/\alpha$. Since $K_{N_\nu} \to 1$ as $\nu \to \infty$, there exists a positive integer ν_0 such that if $\nu \geq \nu_0$, then

$$\frac{r_\nu}{2} \leq S_\nu \leq R_\nu.$$

Hence we may choose $r = S_\nu$ $(\nu \geq \nu_0)$ in Lemma 3.3 to obtain by Lemma 3.1 and an obvious variant of (3.12) that

$$\begin{aligned} \log M(S_\nu, F) &\geq \log M\left(\frac{S_\nu}{r_\nu}, f(z, c_\nu)\right) - \{4 + o(1))\}c_\nu \\ &= \frac{c_\nu}{1 - K_{N_\nu}/\alpha} - \{4 + o(1))\}c_\nu \\ &= \left\{\frac{\alpha}{\alpha - K_{N_\nu}} - 4 - o(1)\right\} c_\nu, \quad \nu \to \infty, \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} m_p^+(S_\nu, F) &\geq m_p^+\left(\frac{S_\nu}{r_\nu}, f(z, c_\nu)\right) - \{4 + o(1))\}c_\nu \\ &= \frac{1}{2\pi}\left\{\frac{c_\nu}{(1 - K_{N_\nu}/\alpha)^{\frac{p-1}{p}}}\right\} - \{4 + o(1)\}c_\nu \\ &= \left\{\frac{1}{2\pi}\left(\frac{\alpha}{\alpha - K_{N_\nu}}\right)^{\frac{p-1}{p}} - 4 - o(1)\right\} c_\nu, \quad \nu \to \infty. \end{aligned} \tag{3.16}$$

Thus we conclude by (3.10), (3.11), (3.13), (3.14), and (3.5) (iii) that for any $\varepsilon > 0$,

$$\begin{aligned}
&\overline{\lim_{r\to\infty}} \frac{\log M(r,F)}{T(r,F)^{\frac{1}{2}+\varepsilon}T(\alpha r,F)^{\frac{1}{2}-\varepsilon}} \geq \overline{\lim_{\nu\to\infty}} \frac{\log M(R_\nu,F)}{T(R_\nu,F)^{\frac{1}{2}+\varepsilon}T(\alpha R_\nu,F)^{\frac{1}{2}-\varepsilon}} \\
&\geq \overline{\lim_{\nu\to\infty}} \frac{\{1/8\sqrt{2}-o(1)\}c_\nu^{\frac{1}{2}}N_\nu^{\frac{1}{2}}}{[\{1+o(1)\}c_\nu]^{\frac{1}{2}+\varepsilon}[\{1+o(1)\}N_\nu \log\alpha]^{\frac{1}{2}-\varepsilon}} \\
&= \overline{\lim_{\nu\to\infty}} \frac{(N_\nu c_\nu^{-1})^\varepsilon}{8\sqrt{2}(\log\alpha)^{\frac{1}{2}-\varepsilon}} = \infty,
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
&\overline{\lim_{r\to\infty}} \frac{m_p^+(r,F)}{T(r,F)^{\frac{p+1}{2p}+\varepsilon}T(\alpha r,F)^{\frac{p-1}{2p}-\varepsilon}} \\
&\geq \overline{\lim_{\nu\to\infty}} \frac{m_p^+(R_\nu,F)}{T(R_\nu,F)^{\frac{p+1}{2p}+\varepsilon}T(\alpha R_\nu,F)^{\frac{p-1}{2p}-\varepsilon}} \\
&\geq \overline{\lim_{\nu\to\infty}} \frac{\{1/16\sqrt{2}\pi-o(1)\}c_\nu^{\frac{p+1}{2p}}N_\nu^{\frac{p-1}{2p}}}{[\{1+o(1)\}c_\nu]^{\frac{p+1}{2p}+\varepsilon}[\{1+o(1)\}N_\nu\log\alpha]^{\frac{p-1}{2p}+\varepsilon}} \\
&= \overline{\lim_{\nu\to\infty}} \frac{(N_\nu c_\nu^{-1})^\varepsilon}{16\sqrt{2}\pi(\log\alpha)^{\frac{p-1}{2p}-\varepsilon}} = \infty.
\end{aligned} \tag{3.18}$$

Thus such an F satisfies conclusion (a) of Theorem 3.2. Noting $\alpha S_\nu = R_\nu$, we conclude by (3.10) and (3.15) that

$$\begin{aligned}
&\overline{\lim_{r\to\infty}} \frac{\log M(r,F)}{T(\alpha r,F)} \geq \overline{\lim_{\nu\to\infty}} \frac{\log M(S_\nu,F)}{T(R_\nu,F)} \geq \overline{\lim_{\nu\to\infty}} \frac{\{\frac{\alpha}{\alpha-K_{N_\nu}}-4-o(1)\}c_\nu}{\{1+o(1)\}c_\nu} \\
&= \frac{\alpha}{\alpha-1}-4 = \frac{4-3\alpha}{\alpha-1}.
\end{aligned} \tag{3.19}$$

We also have by (3.10) and (3.16) that

$$\begin{aligned}
&\overline{\lim_{r\to\infty}} \frac{m_p^+(r,F)}{T(\alpha r,F)} \geq \lim_{\nu\to\infty} \frac{m_p^+(S_\nu,F)}{T(R_\nu,F)} \\
&\geq \overline{\lim_{\nu\to\infty}} \frac{\{\frac{1}{2\pi}(\frac{\alpha}{\alpha-K_{N_\nu}})^{\frac{p-1}{p}}-4-o(1)\}c_\nu}{\{1+o(1)\}c_\nu} \\
&= \frac{1}{2\pi}\left(\frac{\alpha}{\alpha-1}\right)^{\frac{p-1}{p}} - 4 \geq \left(\frac{1}{\alpha-1}\right)^{\frac{p-1}{p}}\left\{\frac{1}{2\pi} - 4(\alpha-1)^{\frac{p-1}{p}}\right\}.
\end{aligned} \tag{3.20}$$

Thus such an F satisfies conclusions (b) of Theorem 3.2.

Finally, our functions $F_i(z)$ $(i = 1, 2, 3)$ are to be constructed just as $F(z)$ is, with carefully chosen sequences r_n, c_n, and N_n satisfying (3.1) and (3.5).

We first construct $F_1(z)$ by choosing r_n, c_n, and N_n as follows. We set $r_1 = c_1 = 1$ and $N_1 = 512$. Suppose that r_n, c_n, and N_n have been chosen for $n < \nu$. Then we choose r_ν so large that

$$\text{(3.21) (i)} \qquad r_\nu > 4r_{\nu-1} \quad \text{and} \quad \varphi\left(\frac{r_\nu}{2}\right) > \nu^3 N_{\nu-1},$$

where $\varphi(r)$ is the function occurring in the hypothesis of Theorem 3.2. This is possible since $\varphi(r) \to \infty$ as $r \to \infty$. We then set

$$\text{(3.21) (ii)} \qquad N_\nu = \left[\varphi\left(\frac{r_\nu}{2}\right) \log r_\nu\right],$$

and

$$\text{(3.21) (iii)} \qquad c_\nu = N_\nu / 512\nu.$$

We next construct $F_2(z)$ by choosing r_n, c_n, and N_n as follows. Let $r_1 = c_1 = 1$ and let $N_1 = 512$. Suppose that r_n, c_n, and N_n have been chosen for $n < \nu$. Then we choose r_ν so large that

$$\text{(3.22) (i)} \qquad \log r_\nu > \nu N_{\nu-1} r_{\nu-1}.$$

We set

$$\text{(3.22) (ii)} \qquad c_\nu = (\log r_\nu)^2,$$

and

$$\text{(3.22) (iii)} \qquad N_\nu = \left[r_\nu^\lambda c_\nu^2\right].$$

Finally we construct $F_3(z)$ by choosing r_n, c_n, and N_n as follows. For $n \geq 1$, we set

$$\text{(3.23) (i)} \qquad r_n = 4^{n-1},$$

$$\text{(ii)} \qquad c_n = (n!)^4,$$

and

$$\text{(iii)} \qquad N_n = 512 n c_n.$$

It is easy to show that the choices of sequences in (3.21), (3.22), and (3.23) all satisfy (3.1) and (3.5). Hence $F_i(z)$ $(1 \le i \le 3)$ are entire functions satisfying the conditions (a) and (b) of Theorem 3.2.

It remains to estimate the growth properties of $F_i(z)$. From Lemma A(c), Lemma B(a), and (3.5) (iii) we deduce for $i = 1, 2$ and $r_\nu/2 \le r < r_{\nu+1}/2$ that

$$T(r, F_i) \le T\left(\frac{r}{r_\nu}, e^{-4c_\nu} P_{N_\nu}(z, c_\nu)\right) + 0(\nu N_{\nu-1} \log r) = 0(N_\nu \log r), \quad r \to \infty.$$

Therefore by (3.21) (ii) and (3.22) (ii), (iii), we have

$$T(r, F_1) = 0([\varphi(r_\nu/2) \log r_\nu] \log r) = 0(\varphi(r)(\log r)^2), \quad r \to \infty,$$

and

$$T(r, F_2) = 0(\left[r_\nu^\lambda c_\nu^2\right] \log r) = 0(r^\lambda (\log r)^5), \quad r \to \infty.$$

Hence $F_1(z)$ satisfies the required growth condition and $F_2(z)$ has at most order λ. On the other hand, note that the Maclaurin coefficients a_n of $f(z, c_\nu)$ are all positive. Since

$$\begin{aligned} f(z, c_\nu)^{(n)} &= \left\{\exp\left(\frac{c_\nu}{1-z}\right)\right\}^{(n)} = \left\{\frac{c_\nu}{(1-z)^2} \exp\left(\frac{c_\nu}{1-z}\right)\right\}^{(n-1)} \\ &= \cdots = \frac{n! c_\nu}{(1-z)^{n+1}} \exp\left(\frac{c_\nu}{1-z}\right) + \cdots, \end{aligned}$$

we estimate for all n that

$$a_n \ge c_\nu e^{c_\nu}.$$

Hence we have

$$\begin{aligned} &\log M(e r_\nu, F_2) = \log F_2(e r_\nu) \ge \log P_{N_\nu}(e, c_\nu) - 4c_\nu \\ &\ge \log(a_{N_\nu} e^{N_\nu}) - 4c_\nu \ge (1 - o(1)) N_\nu = (1 - o(1)) r_\nu^\lambda c_\nu^2, \quad \nu \to \infty. \end{aligned}$$

Thus $F_2(z)$ has order λ and satisfies the growth condition of the theorem.

Next we want to show that $F_3(z)$ has infinite order. By virtue of (3.13) and (3.23), we have

$$\log M(R_\nu, F_3) \ge \left(\frac{1}{8\sqrt{2}} - o(1)\right) c_\nu^{\frac{1}{2}} N_\nu^{\frac{1}{2}} \ge \left(\frac{1}{8\sqrt{2}} - o(1)\right) (\nu!)^4, \quad \nu \to \infty,$$

and

$$R_\nu \le r_\nu = 4^{\nu-1}.$$

Hence for any positive number ℓ,

$$\overline{\lim_{\nu\to\infty}} \frac{\log M(R_\nu, F_3)}{R_\nu^\ell} \ge \overline{\lim_{\nu\to\infty}} \frac{\left(\frac{1}{8\sqrt{2}} - o(1)\right)(\nu!)^4}{4^{\ell(\nu-1)}} = \infty.$$

Thus F_3 has infinite order.

Finally we need to show that $F_2(z)$ satisfies (c) of Theorem 3.2. By (3.10), (3.13), (3.14), and (3.22) we have

$$\overline{\lim_{r\to\infty}} \frac{\log M(r, F_2)}{T(r, F_2) r^{\frac{\lambda}{2}}} \ge \overline{\lim_{\nu\to\infty}} \frac{\log M(R_\nu, F_2)}{T(R_\nu, F_2) R_\nu^{\frac{\lambda}{2}}} \ge \overline{\lim_{\nu\to\infty}} \frac{\left(\frac{1}{8\sqrt{2}} - o(1)\right) c_\nu^{\frac{1}{2}} N_\nu^{\frac{1}{2}}}{(1+o(1)) c_\nu R_\nu^{\frac{\lambda}{2}}}$$
$$\ge \overline{\lim_{\nu\to\infty}} \frac{c_\nu^{\frac{1}{2}} (r_\nu^\lambda c_\nu^2)^{\frac{1}{2}}}{8\sqrt{2} c_\nu r_\nu^{\frac{\lambda}{2}}} = \overline{\lim_{\nu\to\infty}} \frac{c_\nu^{\frac{1}{2}}}{8\sqrt{2}} = \infty,$$

and

$$\overline{\lim_{r\to\infty}} \frac{m_p^+(r, F_2)}{T(r, F_2) r^{\frac{p-1}{2p}\lambda}} \ge \overline{\lim_{\nu\to\infty}} \frac{m_p^+(R_\nu, F_2)}{T(R_\nu, F_2) R_\nu^{\frac{p-1}{2p}\lambda}}$$
$$\ge \overline{\lim_{\nu\to\infty}} \frac{\left(\frac{1}{16\sqrt{2\pi}} - o(1)\right) c_\nu^{\frac{p+1}{2p}} N_\nu^{\frac{p-1}{2p}}}{(1+o(1)) c_\nu R_\nu^{\frac{p-1}{2p}\lambda}}$$
$$\ge \overline{\lim_{\nu\to\infty}} \frac{c_\nu^{\frac{p+1}{2p}} (r_\nu^\lambda c_\nu^2)^{\frac{p-1}{2p}}}{16\sqrt{2\pi} c_\nu r_\nu^{\frac{p-1}{2p}\lambda}} = \overline{\lim_{\nu\to\infty}} \frac{c_\nu^{\frac{p-1}{2p}}}{16\sqrt{2\pi}} = \infty.$$

The proof of Theorem 3.2 is now complete.

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